**Chapter - 6**

**Gradient, Divergence and Curl**

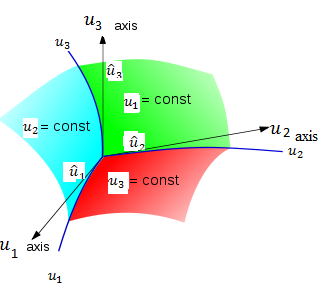
1. **The Gradient Vector: grad**

Consider a room in which the temperature is given by a scalar field, at each point  (assume that the temperature does not change over time.). At each point in the room, the gradient of  at that point will show the direction in which the temperature rises most quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

A vector field, called the gradient, written: or , where is called ‘del’, can be associated with a scalar field .

At every point, the direction of the vector field () is

* orthogonal to the scalar field contour and
* in the direction of the maximum rate of change of .



The gradient of a scalar function is given by

where are scale factors and are the unit vectors along , For cartesian coordinates , for cylindrical coordinates and and for spherical coordinates

**Example 1.1.** Find the gradient of the following scalar functions:

(a) at the point

For Cartesian coordinates,

At the point

(b) at the point

For cylindrical coordinates,

At the point

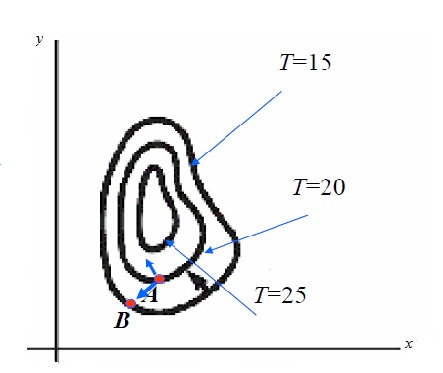
(c) at the point

For cylindrical coordinates,

.

At the point .

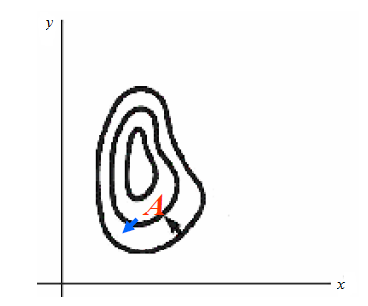
1. **Directional Derivatives:**

Consider the temperature at various points of a heated metal plate. Some contours for are shown in the following diagram.

We are interested in how changes from one point to another. The rate of change of in the direction specified by is given by , examples of **directional derivative.**

In general, for a given function , **the directional derivative** in the direction of a unit vector is the **gradient vector** at a point

* magnitude the largest directional derivative, and
* pointing in the direction in which this largest directional derivative occurs, is known as the **gradient vector**.



Hence, the component of in the direction of a vector is equal to and it is called the directional derivative of in the direction of.

**Example 2.1** Find the directional derivative of at the point in the direction

**Solution:**

At the point ,

Now, the unit vector in the direction of is

Then the required directional derivative is,

**Example 2.2** Find the directional derivative of at the point in the direction .

**Solution:**

At the point ,

Then the required directional derivative is,

**Example 2.3** Find the directional derivative of at the point in the direction .

**Solution:**

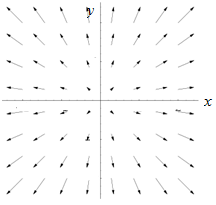
At the point ,

Then the required directional derivative is,

1. **The divergence and curl of a vector function**

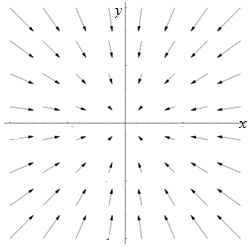
Consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air is cooled and thus contracting, the divergence of the velocity has a negative value.

The divergence of a vector field is relatively easy to understand intuitively. Imagine that the vector field **A**pictured below gives the velocity of some fluid flow. It appears that the fluid is exploding outward from the origin.



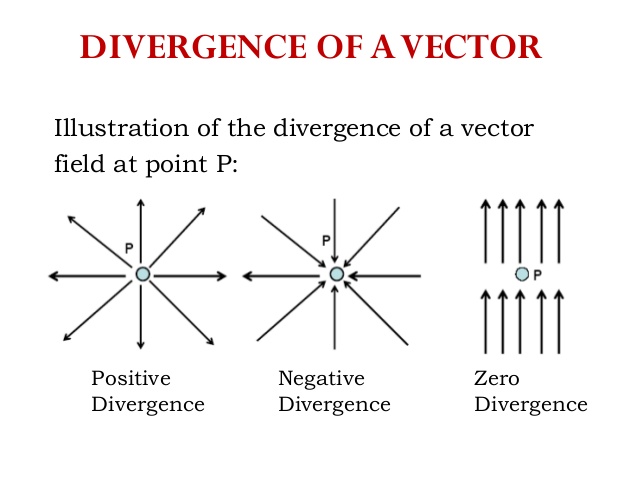
This expansion of fluid flowing with velocity field is captured by the divergence of , which we denote or mathematically . The divergence of the above vector field is positive since the flow is expanding.

In contrast, the below vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative.

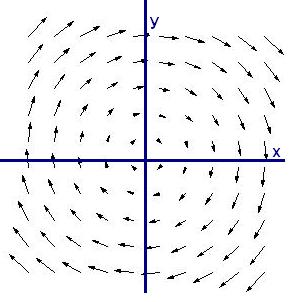


Lastly, a **solenoidal vector field** (also known as an **incompressible vector field**, a **divergence-free vector field**) is a vector field  with divergence zero at all points in the field. That is

Hence the illustration of the divergence of a vector field at any point is given below:



The curl of a vector field captures the idea of how a flow may rotate. Imagine that the below vector field F represents fluid flow. The vector field indicates that the fluid is circulating around a central axis. This rotation of fluid flowing with velocity field is captured by the curl of , which we denote or mathematically.



If then is called **conservative or irrotational**.

Now, let is a vector. The divergence and the curl is given by

and

* The derivation can be found in advanced calculus books.

**Example 3.1** Determine divergence and curl. Also check each of the following vector fields solenoidal, conservative or both.

(a)

is not solenoidal or conservative.

(b)

is not solenoidal or conservative.

(c)

is conservative but not solenoidal.

**Example 3.2** Test whetheris a conservative force field. If conservative, find the scalar potentialsuch that. Hence find the work done in moving an object in this field from to

**Solution:** We know for conservative force field

Hence is conservative force field.

Let be a scalar potential of **,** i.e.

, is a constant.

Now, work done,

**Example 3.3** Test whetheris a conservative force field. If conservative, find the scalar potentialsuch that. Hence find the work done in moving an object in this field from to

**Solution:** We know for conservative force field

Hence is conservative force field.

Letbe a scalar potential of **,** i.e.

, is a constant.

Now, work done,

**Example 3.4** Test whetheris a conservative force field. If conservative, find the scalar potentialsuch that. Hence find the work done in moving an object in this field from to

**Solution:** We know for conservative force field

Hence is conservative force field.

Letbe a scalar potential of **,** i.e.

, is a constant.

Now, work done,

1. **Laplacian operator**

The Laplacian of a scalar function is defined as the divergence of the gradient of that function.

The differential operator is called Laplacian operator, and

is called Laplace Equation.

The Laplacian of a scalar function in different coordinate system are defined as follows:

In Cartesian coordinates

in Cylindrical coordinates and

in Spherical coordinates

**Example 4.1** Find the Laplacian of the scalar function

**Solution:** In Cartesian co-ordinates we know the Laplacian is

**Example 4.2** Find the Laplacian of the scalar function

**Solution:** In Cylindrical coordinates we know the Laplacian is

.

**Example 4.3** Find the Laplacian of the scalar function

**Solution:** In Spherical coordinates we know the Laplacian is,

**Sample exercise - 6.1**

**1**. Find the gradient of the following scalar functions at the indicated point:

**(a)** at the point Ans:

**(b)** at the point Ans:

**(c)** at the point

Ans:

**2. (a)** Find the directional derivative (D. D.) of at the point in the direction Ans: and D. D. is

**(b)** Find the directional derivative (D. D.) of at the point in the direction . Ans: D.D. is

**(c)** Find the directional derivative (D. D.) of at the point in the direction. Ans: D. D. is

**3.** Determine divergence and curl. Also check each of the following vector fields solenoidal, conservative or both.

**(a)**

Ans: and

**(b)** Ans: and

**(c)** Ans: and

**4. (a)** Test whether is a conservative force field. If conservative, find the scalar potentialsuch that. Hence find the work done in moving an object in this field from to Ans:

**(b)** Test whetheris a conservative force field. If conservative, find the scalar potentialsuch that. Hence find the work done in moving an object in this field from to Ans:

**(c)** Test whether is a conservative force field. If conservative, find the scalar potential such that. Hence find the work done in moving an object in this field from to Ans:

**5.** Find the Laplacian of the following scalar functions:

**(a)**  Ans:

**(b)** Ans:

**(c)** Ans:

**(d)** Ans:

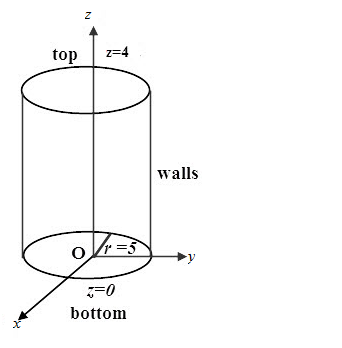
**MATLAB for** **Gradient, divergence and curl**

|  |  |
| --- | --- |
| Find the gradient of the following scalar functions | >>syms x y z  >>gradient(3/(x^2 + z^2), [x, y, z]) |
| Determine divergence of | >>syms x y z  >>divergence([x^2, 2\*y^2, 0], [x, y, z]) |
| Determine curl of | >>syms x y z  >>curl([x, 2\*y^2, 3\*z^3], [x, y, z]) |
| Find the Laplacian of | >> syms x y z  >> laplacian(4\*x\*y^2\*z^3) |

**Gauss’s-Divergence Theorem and Stoke’s Theorem**

1. **Some Prerequisite Example**

**Example 1.1*.***  For a vector function find surface integral for the circular cylindrical region enclosed by



**Solution**

i. Top face: and

ii. Bottom face: and

iii. Walls face: and

Total,

**Example 1.2*.*** Evaluate http://tutorial.math.lamar.edu/Classes/CalcIII/TISphericalCoords_files/empty.gif where *E* is the upper half of the sphere

http://tutorial.math.lamar.edu/Classes/CalcIII/TISphericalCoords_files/empty.gif.

**Solution**

Since we are taking the upper half of the sphere the limits for the variables are,

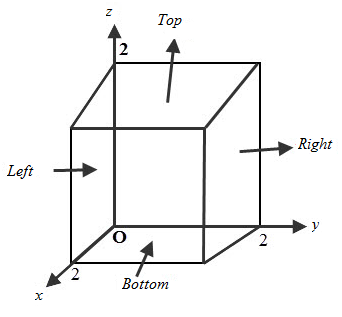
The integral is then,

1. **Gauss’s divergence theorem**

**Statement:** The surface integral of the normal component of a vector function taken arround a closed surfce is equal to the integral of the divergence of taken over the volume enclosed by the surface .

Mathematically,

**Example 2.1.** For the vector field , verify the divergence theorem by computing **(a)** the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to units each and parallel to the Cartesian axes,**(b)** the integral of over the cube’s volume.



**Solution**

We first evaluate the surface integral over the six faces.

i. Front face:

ii. Back face:

iii. Right face:

iv. Left face:

v. Top face:

vi. Bottom face:

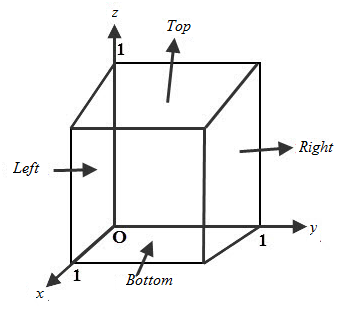
Adding the above six values, we have

Now the divergence of is

Hence

which is the same as the result of the closed surface integral. The divergence theorem is therefore verified.

**Example 2.2.** Given verify the divergence theorem over a cubeone unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin.



**Solution**

We first evaluate the surface integral over the six faces.

i. Front face:

ii. Back face:

iii. Right face:

iv. Left face:

v. Top face:

vi. Bottom face:

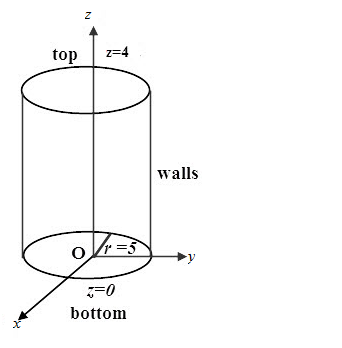
Adding the above six values, we have

Now the divergence of is

Hence

which is the same as the result of the closed surface integral. The divergence theorem is therefore verified.

**Example 2.3.** For a vector function verify for the circular cylindrical region enclosed by



**Solution**

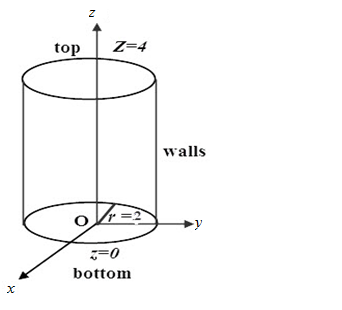
i. Top face: and

ii. Bottom face: and

iii. Walls face: and

Total, (verified).

**Example 2.4** A vector field verify the divergence theorem for the cylindrical region enclosed by and



**Solution**

.

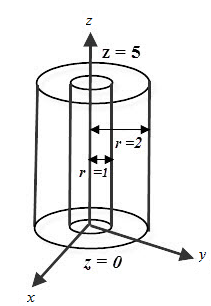
i. Top face: and

ii. Bottom face: and

iii. Walls face: and and

Total

**Example 2.5** A vector field exists in the region between two concentric cylindrical surfaces defined by and , with both cylinders extending between and . Verify the divergence theorem by evaluating the following:(a) and (b) .



**Solution:**

.

.

i. Top face: and

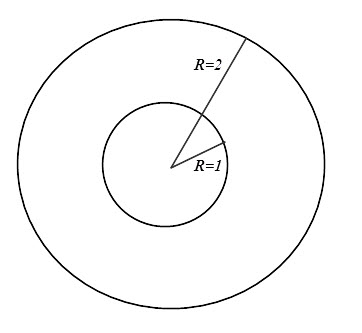
ii. Bottom face: and

iii. Outside surface: and

iv. Inside surface: and

Adding all surface, .

**Example 2.6** For the vector field evalute both sides the divergence theorem for the region enclosed between spharical shells defined by and .



**Solution**

At the outer surface and we get

.

At the inner surface and we get

.

Total

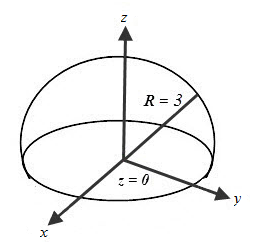
Again,

Now, Outer sphere:

Inner sphere:

Total

**Example 2.7** Find over the surface of a hemispherical region that is the top half of a sphere of radius centered at with its flat base coinciding with the plane. Also verify divergence theorem where .



**Solution**

Given that

In spherical coordinates

Over the hemisphere surface and we get

.

Now,

and

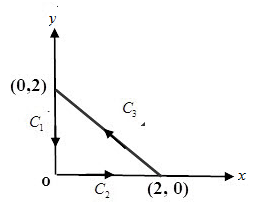
So,

**Sample exercise - 6.2**

1. For the vector field, verify the divergence theorem by computing **(a)** the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to units each and parallel to the Cartesian axes, **(b)** the integral of over the cube’s volume.
2. For a vector function verify for the circular cylindrical region enclosed by Ans:
3. A vector field exists in the region between two concentric cylindrical surfaces defined by and , with both cylinders extending between and . Verify the divergence theorem by evaluating the following: (a) and (b) . Ans:
4. Find over the surface of a hemispherical region that is the top half of a sphere of radius centered at with its flat base coinciding with the plane. Also verify divergence theorem. where . Ans:
5. For the vector field evaluate both sides of the divergence theorem for the region enclosed between spherical shells defined by and . Ans:
6. **Stokes’s Theorem:**

Let be the open surface (two-sided) and be the closed boundary of , the vector field is continuous on . Then

**Example 3.1.** Assume that a vector field , (a) find around the triangular contour, (b) find over triangular arc, (c) verify Stokes’s theorem and (d) can be expressed as gradient of a scalar? Explain.



**Solution**

(a)

Path ;

Path ;

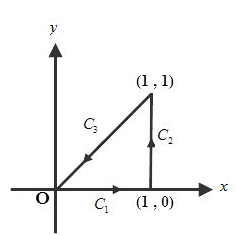
Path ;

Total,

Now,

(c) Stokes’s theorem is verified. (d) No,

**Example 3.2.** Assume that a vector field, (a) find around the triangular contour, (b) find over triangular arc.



**Solution**

(a)

Path ;

Path ;

Path ;

Total,

Now,

Stokes’s theorem is verified.

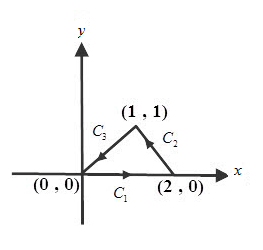
**Example 3.3.** Repeat **Ex3.2.** for the contour shown below.

**Solution**

(a)

Path ;

Path ;

****

Path ;

Total,

Now,

Stokes’s theorem is verified.

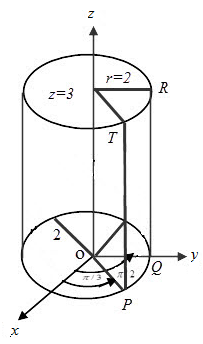
**Example 3.4.** For vector field , verify Stokes’s theorem for a segment of a cylindrical surface defined by , and .

**Solution**

Stokes’s theorem states that

**LHS**: With having only a component use of the expression for in cylindrical coordinates,

The integral of over the specified surface is



**RHS**: The surface *S* is bounded by contour shown in figure above. The direction of is chosen so that it is compatible with the surface normal by the right-hand rule. Hence,

where **,**, and are the field along segments and respectively. Over segment the dot product of and is zero, and the same is true for segment . Over segment , ; hence, For the last segment, and . Hence,

which is the same as the result obtained by evaluating the left-hand side of Stokes’s equation.

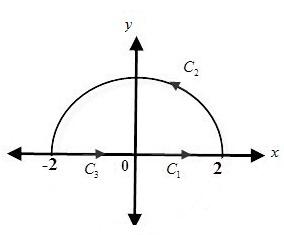
**Example 3.5.** Assume that a vector field , (a) find over the semicircular contour, and (b) find over the surface of the semicircle.

**Solution**

(a)

Path ;

Path ;



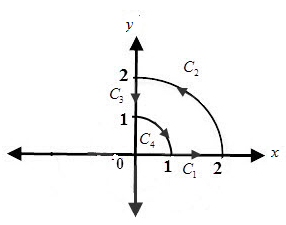
Path ;

Total,

Now,

Stokes’s theorem is verified.

**Example 3.6.** Repeat **Ex3.5.** for the contour shown below.

****

**Solution**

(a)

Path ;

Path ;

Path ;

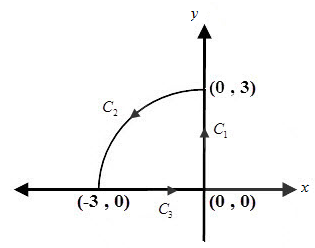
Path ;

Total,

Now,

Stokes’s theorem is verified.

**Example 3.7.** Assume that a vector field , (a) find over the path comprising a quarter section of a circle, and (b) find over the surface of the quarter section.



**Solution**

(a)

Path ;

Path ;

Path ;

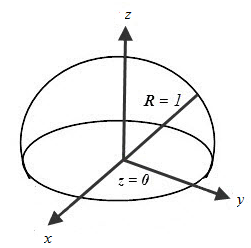
Total,

Now,

Stokes’s theorem is verified.

**Example 3.8.** Verify Stokes’s theorem for the vector field, by evaluating it on the hemisphere of unit radius.

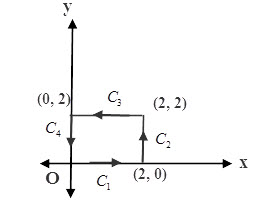
**Solution**

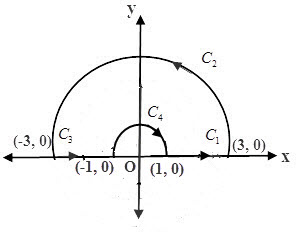
and ,

Again, ,

Stokes’s theorem is verified.

**Sample exercise – 6.3**

1. Verify Stokes’s theorem for the vector field by evaluating it on the hemisphere of radius 2. Ans:
2. For vector field , verify Stokes’s theorem for a segment of a cylindrical surface defined by , and . Ans:
3. Assume that a vector field , (a) find around the rectangular contour, (b) find over rectangular arc, (c) verify Stokes’s theorem and (d) can be expressed as gradient of a scalar? Explain. Ans: 
4. Assume that a vector field,, (a) find over the semicircular contour shown below, and (b) find over the surface of the semicircles. Ans:



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